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Local Thermodynamic Equilibrium for some Stochastic Models of Hamiltonian Origin

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We consider a class of 1-D stochastic models that are realizations of Hamiltonian models of heat conduction and prove that in the infinite volume limit local thermodynamic equilibrium is attained with linear energy profile.

KEY WORDS: local thermodynamic equilibrium, Hamiltonian systems, scaling limits, random walks, Martingales, interacting particle systems

For a system in a nonequilibrium steady state, a fundamental issue is whether or not it is locally in thermodynamic equilibrium (LTE), with a well defined notion of local temperature. This has remained a challenging mathematical problem for even relatively simple models.

In this paper, we prove LTE for a class of 1-D models that came about as stochastic realizations of certain mechanical models of energy transport. These mechanical models were introduced and studied numerically in Refs. 9, 13. In Ref. 4, they were simplified slightly, stochastic realizations were proposed, and the results in the present paper were announced.

We begin with a description of the mechanical models that motivate this work, giving the slightly simplified version in Ref. 4. The physical domain of this Hamiltonian system is a linear array of N identical boxes separated by walls with holes to allow passage between adjacent boxes. See Fig. 1. The two ends of the chain are connected to infinite heat reservoirs. Inside each box there is a rotating disk nailed down at its center, around which it turns freely. The kinetic energies of these N disks serve to mark the energy levels along the chain. Rotating disks aside, the only other agent in the system is a single point particle (called a *tracer*)

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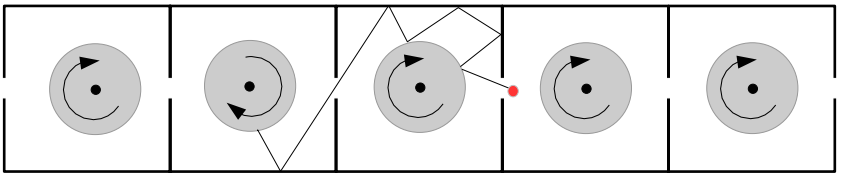

Fig. 1. Tracer particle and linear array of rotating disks.

which serves to mediate the redistribution of energy. Except at collisions with the disks, the tracer moves freely in uniform motion along the chain, making elastic collisions with the walls of the boxes. When it collides with a disk, the interaction is that of "sticky reflection": If the angular velocity of the disk prior to collision is ω, and v_n and v_t are the normal and tangential components of the velocity of the approaching tracer relative to the point of impact, then the values of ω', v_n' and v_t' after the collision are given by the following energy and angular momentum conservation laws:

$$v_n' = -v_n, \quad v_t' = v_t - \frac{2\varepsilon}{1+\varepsilon}(v_t - \omega), \quad \omega' = \omega + \frac{2}{1+\varepsilon}(v_t - \omega).$$

These rules of interaction are taken verbatim from Refs. 9, 13. They were in fact used earlier in Ref. 14. When the tracer reaches the ends of the chain, it is replaced by a new tracer carrying a kinetic energy characteristic of the heat reservoir. More precisely, suppose the tracer exits the left hole of the leftmost box. Instantaneously, a new tracer is injected into the system from the same hole, at a random angle, and carrying a kinetic energy drawn i.i.d. from an exponential distribution with mean T_L, the temperature of the left bath. An analogous set of rules holds when the tracer exits the hole at the far right. The temperatures T_L and T_R of the left and right baths are set unequal, thereby forcing the system out of equilibrium.

In Fig. 1, the individual boxes in the chain are shown as squares. That need not be the case. Indeed, concave walls induce better scattering of the tracer and leads generally to better mixing properties of the system.

When the system described above is in equilibrium (i.e., when $T_L = T_R$), a family of Gibbs measures is preserved. For purposes of the present discussion, let us assume that when $T_L \neq T_R$, this system has a unique steady state to which all initial conditions converge. We now fix $T_L \neq T_R$, and focus on a subsystem of ℓ consecutive disks at a particular location of the chain, say $\frac{1}{3}$ of the length of the chain from the left. If $\ell \ll N$ where N is total system size, then the temperature gradient across this subsystem should be rather small. It should, therefore be close to being in equilibrium, i.e. the projection of the invariant measure of the full system onto this subsystem should be close to one of the Gibbs measures. If this is true, we say the system is locally in equilibrium, or that it approaches LTE in the infinite-volume limit. See Ref. 7 for the physics of nonequilibrium systems.

Of interest also is the temperature profile along the chain. Fourier's Law predicts that heat flux is proportional to temperature gradient times conductivity (which may be temperature dependent). See the review papers[3,11] on Fourier's Law; for recent work see e.g. Refs. 1, 2.

Techniques for proving LTE for systems defined by purely deterministic microscopic laws are, however, not on the horizon. Stochastic models have, in the past, been introduced to gain insight into the statistical mechanics of Hamiltonian systems that are otherwise intractable; see e.g. Refs. 8, 10, 12. In the same spirit, the authors of Ref. 4 proposed stochastic realizations of the models described above in the hope of capturing the essence of the Hamiltonian dynamics.

In the stochastic realizations proposed in Ref. 4, one considers random variables $(\xi_1, \xi_2, \ldots, \xi_N)$ and η representing the energies of the N disks and the tracer, and the system is assumed to be Markovian, with the times for energy sharing and tracer movements signaled by exponential clocks. There are two other simplifying assumptions. The first is the rule of interaction between tracer and disk: In the stochastic model we pool together the energies of the tracer and disk, and divide it randomly into two parts, giving one part to the tracer and the other to the disk. This is to simulate the fact that in the Hamiltonian model, different conditions prior to a collision, such as the angle at which the tracer approaches the disk, lead to different outcomes of energy repartition. If the system is sufficiently mixing, these angles appear quite random. The second simplifying feature is that the tracer is assumed of have no memory of its past; it jumps with equal probability to the left and right when the clock goes off. We require, however, that the clocks ring at rates proportional to the square root of the (kinetic) energy of the tracer, i.e. its speed. This is because in the Hamiltonian model, time to collision is halved if the speed of the tracer with the same trajectory is doubled. It has been demonstrated in Refs. 4, 5 that this aspect of the dynamics must be respected if the stochastic models are to retain some of the character of their Hamiltonian counterparts. A detailed description of the stochastic models considered in this paper is given in Sec. 1.1.

To prove LTE for these models, we adapted certain ideas of duality for particle systems invented by Spitzer. In this duality, local equilibrium in the original process corresponds to asymptotic independence of random walks in certain particle systems. Once the latter is proved, linearity of mean energy profile is an immediate corollary. A similar idea is used in Ref. 8 to treat a class of models now known as the KMP models. KMP models are also stochastic realizations of Hamiltonian systems, in their case nearest neighbor coupling of Hamiltonian systems with a single degree of freedom. We learned about this particular dual process from Ref. 8. Indeed a part of our proof can be seen as an adaptation of theirs.

Hydrodynamics of stationary nonequilibrium states for a large class of stochastic lattice gas models were studied in Ref. 6. It is likely that entropy meth-

ods can be used to give results for a range of models including those considered in this paper. We have chosen to use the duality method here because it is simpler and equally effective for this particular model.

1. SETTING AND RESULTS

1.1. Description of Model

We consider in this paper a stochastic process on a (finite) 1-D lattice $\{1, 2, \ldots, N\}$, $N \in \mathbb{Z}^+$. A rough description of the model is as follows: Sites 1 and N are in contact with heat baths, which we may think of as located at 0 and $N + 1$. The temperatures of the left and right baths are T_L and T_R respectively. At each of the N sites, there is a stored energy, denoted by ξ_i. Energy transfer among the different sites is mediated by a tracer particle which moves from site to site. Each time the tracer visits a bath, it returns with an energy characteristic of the bath. The object of our investigation is the steady state distribution of the energy configuration $(\xi_1, \ldots, \xi_N)$ as $N \to \infty$.

We proceed now to a precise definition of our models, first giving a description in words before writing down a generator. Given bath temperatures $T_L, T_R > 0$ and lattice size $N \in \mathbb{Z}^+$, we consider a continuous-time Markov jump process $\mathbf{X}_t$ defined by random variables

$$\mathbf{X}_t := (\alpha_t, \eta_t;\ \xi_{1,t}, \xi_{2,t}, \ldots, \xi_{N,t}), \quad t \geq 0,$$

where $\alpha_t \in \{1, 2, \ldots, N\}$ denotes the location of the tracer at time t, $\eta_t \in [0, \infty)$ denotes the (kinetic) energy of the tracer at time t, and $\xi_{i,t} \in [0, \infty)$ the stored energy at site i at time t. Given $(\alpha_t, \eta_t, \xi_{1,t}, \ldots, \xi_{N,t})$, the next action takes place at time $t' > t$ where $t' - t$ is distributed according to an exponential law with mean $\sqrt{\eta_t}$. More precisely, $(\alpha_s, \eta_s, \xi_{1,s}, \ldots, \xi_{N,s}) = (\alpha_t, \eta_t, \xi_{1,t}, \ldots, \xi_{N,t})$ for all $s \in [t, t')$, and at time t', the following take place:

(1) *Mixing of energies*: If $\alpha_t = i$, then

$$\xi_{i,t'} = p(\xi_{i,t} + \eta_t) \quad \text{and} \quad \eta_{t'} = (1 - p)(\xi_{i,t} + \eta_t)$$

where $p \in [0, 1]$ is uniformly distributed – except in the cases discussed in (3) below where $\eta_{t'}$ is different.

(2) *Movement of tracer:* If $\alpha_t = i$, $i \neq 1$ or N, then with probability $(\frac{1}{2}, \frac{1}{2})$, $\alpha_{t'} = i \pm 1$. If $\alpha_t = 1$, then $\alpha_{t'} = 1, 2$ with equal probability. Similarly, if $\alpha_t = N$, then $\alpha_{t'} = N, N - 1$ with equal probability.

(3) *Heat bath random variables:* In the cases where $\alpha_t = \alpha_{t'} = 1$ or $\alpha_t = \alpha_{t'} = N$, we think of the tracer as having jumped to the left, resp. right, bath at time t' and instantaneously returned, carrying with it a new energy. Accordingly, we set $\eta_{t'}$ to be a random value distributed according to an exponential law with mean T_L, resp. T_R.

The mixing random variable p in (1), the direction of tracer movement in (2), and the bath energy injected into the system in (3) are chosen independently.

The process above is a continuous-time Markov process on $S := \{1, 2, \ldots, N\} \times [0, \infty)^{N+1}$ defined by the generator L below. Let $\mathbf{X} = (\alpha, \eta, \xi_1, \xi_2, \ldots, \xi_N) \in S$. For $p \in [0, 1]$ and $\xi_L, \xi_R \in (0, \infty)$, let $\mathbf{X}^{\pm,p} = (\alpha^{\pm}, \eta^{\pm}, \xi_1^{\pm}, \xi_2^{\pm}, \ldots, \xi_N^{\pm})$ be given by

$$\xi_j^{\pm} = \begin{cases} \xi_j & \text{if } j \neq \alpha \\ p(\xi_\alpha + \eta) & \text{if } j = \alpha, \end{cases}$$

$$\alpha^+ = \alpha + (1 - \delta_{\alpha,N}),$$

$$\alpha^- = \alpha - (1 - \delta_{\alpha,1}),$$

$$\eta^+ = (1 - p)(\xi_\alpha + \eta)(1 - \delta_{\alpha,N}) + \delta_{\alpha,N}\xi_R,$$

$$\eta^- = (1 - p)(\xi_\alpha + \eta)(1 - \delta_{\alpha,1}) + \delta_{\alpha,1}\xi_L.$$

Then for any continuous function with compact support f on S,

$$Lf(\mathbf{X}) = \int_0^\infty \int_0^\infty \int_0^1 d\xi_L d\xi_R dp \frac{1}{T_L}\frac{1}{T_R} e^{-(\frac{\xi_L}{T_L} + \frac{\xi_R}{T_R})} \times \sqrt{\eta}\left\{\frac{1}{2}(f(\mathbf{X}^{+,p}) + f(\mathbf{X}^{-,p})) - f(\mathbf{X})\right\}.$$

This completes the definition of the process $\mathbf{X}_t$.

We assume throughout that T_L and T_R are arbitrary but fixed; N is fixed in most discussions as well. When there is a need to stress the dependence on N, a superscript "(N)" is added to the relevant notation. For example, $\mathbf{X}_t$ and its invariant measure μ are denoted $\mathbf{X}_t^{(N)}$ and $\mu^{(N)}$, and so on.

1.2. Statement of Results

Proposition 1.1. *Let $T_L = T = T_R$. Then*

$$\mu^e := \left(\frac{1}{N}\sum_{i=1}^N \delta_{\alpha,i}\right) \times \left(d\eta \sqrt{\frac{\beta}{\pi}} \frac{1}{\sqrt{\eta}} e^{-\beta\eta}\right) \times \left(d\xi_1 \cdots d\xi_N \, \Pi_{i=1}^N \beta e^{-\beta\xi_i}\right),$$

$$\beta = \frac{1}{T},$$

is the unique invariant probability measure for $\mathbf{X}_t$.[3]

[3] The values of η and ξ_i are understood to be > 0 throughout.

This paper is concerned primarily with the case where the system is forced out of equilibrium, i.e., when $T_L \neq T_R$.

Proposition 1.2. *For each $T_L, T_R > 0$ and $N \in \mathbb{Z}^+$, $\mathbf{X}_t$ has a unique invariant probability measure μ.*

Unlike the equilibrium case, there is no explicit formula for the density of μ.

Our main result addresses the question of *local thermodynamic equilibrium* (LTE). For $N \in \mathbb{Z}^+$, $x \in (0, 1)$ and $\ell \in \mathbb{N} \cup \{0\}$, let $\mu^{(N)}_{x,\ell}$ denote the projection of $\mu^{(N)}$ onto the $2\ell + 1$ sites $[xN] - \ell, \ldots, [xN] + \ell$ (assuming that $[xN] - \ell > 1$ and $[xN] + \ell < N$). We identify $\mu^{(N)}_{x,\ell}$ with a measure on $\mathbb{R}^{2\ell+1}$ with coordinates $\zeta_{-\ell}, \ldots, \zeta_\ell$, and say $\mathbf{X}^{(N)}_t$ approaches LTE as $N \to \infty$ if for every $x \in (0, 1)$, there exists $\beta(x)$ such that for every $\ell \in \mathbb{N} \cup \{0\}$, $\mu^{(N)}_{x,\ell}$ converges vaguely to the measure

$$\mu_{x,\ell} := d\zeta_{-\ell} \cdots d\zeta_\ell \, \Pi_{i=-\ell}^{\ell} \beta(x) e^{-\beta(x)\zeta_i}. \tag{1}$$

Theorem 1. *As $N \to \infty$, $\mathbf{X}^{(N)}_t$ approaches LTE with $\beta(x)^{-1} = T_L + x(T_R - T_L)$.*

We say the *profile of mean stored energy* as $N \to \infty$ is given by $\Psi : (0, 1) \to \mathbb{R}$ if $\text{graph}(\Psi) = \lim_{N\to\infty} \text{graph}(\Psi^{(N)})$ where $\Psi^{(N)}(\frac{i}{N+1}) = E_{\mu^{(N)}}[\xi_i]$, $1 \le i \le N$.

Corollary 1. *The profile of mean stored energy is linear.*

2. OUTLINE OF PROOFS

This section contains a sketch of the main ideas in this paper. Complete proofs are given in Secs. 3, 4, and 5.

2.1. Discrete-Time Process $\hat{\mathbf{X}}_n$ Associated with $\mathbf{X}_t$

Associated with each $\mathbf{X}_t$ is a simpler process in which the exponential holding time following each jump by the tracer is replaced by the fixed time 1. That is to say, the action – mix followed by jump – takes place at integer times $n = 1, 2, \ldots$. We call this process $\hat{\mathbf{X}}_n$, and use the same letters as before to denote the random variables, i.e.,

$$\hat{\mathbf{X}}_n := (\alpha_n, \eta_n, \xi_{1,n}, \ldots, \xi_{N,n}), \quad n = 0, 1, 2, \ldots.$$

Our plan is to first prove our results for $\hat{\mathbf{X}}_n$, then convert them to results for $\mathbf{X}_t$ by reintroducing holding times.

Lemma 2.1. *For each $T_L, T_R > 0$, $\hat{\mathbf{X}}_n$ has a unique invariant probability measure $\hat{\mu}$. The process is mixing, and for all initial data $(\alpha_0, \eta_0, \xi_{1,0}, \ldots, \xi_{N,0})$, the distribution at time n converges to $\hat{\mu}$ as $n \to \infty$.*

To prove LTE for $\hat{\mathbf{X}}_n$, we need to be able to recognize the measures $\hat{\mu}_{x,\ell}$. The next lemma is used to do that. As before, we identify $\hat{\mu}^{(N)}_{x,\ell}$ with a measure on $\mathbb{R}^{2\ell+1}$.

Lemma 2.2. *Let x and ℓ and β be fixed. If*

$$E_{\hat{\mu}^{(N)}}\left[\Pi_{i=-\ell}^{\ell} \xi_{[xN]+i}^{m_i}\right] \to \left(\Pi_{i=-\ell}^{\ell} m_i!\right) \beta^{-(m_{-\ell}+\cdots+m_\ell)} \quad \text{as } N \to \infty \tag{2}$$

for every $(2\ell+1)$-tuple of nonnegative integers $(m_{-\ell}, \ldots, m_\ell)$, then

$$\hat{\mu}^{N}_{x,\ell} \to \hat{\mu}_{x,\ell} \quad \text{as } N \to \infty$$

where

$$\hat{\mu}_{x,\ell} = d\zeta_{-\ell} \cdots d\zeta_\ell \, \Pi_{i=-\ell}^{\ell} \beta e^{-\beta\zeta_i}.$$

Proving the convergence in (2) with $\beta^{-1} = (1-x)T_L + x T_R$ is therefore the key to proving Theorem 1 for the processes $\hat{\mathbf{X}}_n$.

2.2. Particle Process $\mathbf{Y}_n$ Dual to $\hat{\mathbf{X}}_n$

For each N, we introduce a process $\mathbf{Y}_n$ describing the dynamics of m particles on the lattice $\{\mathrm{L}, 1, 2, \ldots, N, \mathrm{R}\}$. (Let us agree to call the objects here "particles" to distinguish them from the "tracer" in $\hat{\mathbf{X}}_n$.) In $\hat{\mathbf{X}}_n$, energy is transported along the chain by a tracer. In $\mathbf{Y}_n$, the particles move from site to site riding a "vehicle" which plays the same role as the tracer and executes a random walk on $\{1, 2, \ldots, N\}$ in a manner identical to that of the tracer in $\hat{\mathbf{X}}_n$. More precisely,

$$\mathbf{Y}_n := (\gamma_n, Y_{1,n}, Y_{2,n}, \ldots, Y_{m,n}), \quad n = 0, 1, 2, \ldots.$$

Here $\gamma_n \in \{1, \ldots, N\}$ gives the location of the vehicle, and $Y_{j,n} \in \{1, \ldots, N\} \cup \{\mathrm{L}, \mathrm{R}\} \cup \{*\}$ gives the location of the jth particle at time n. The symbol $Y_{j,n} = *$ means particle j is "in transit". Given $\mathbf{Y}_n = (\gamma_n, Y_{1,n}, Y_{2,n}, \ldots, Y_{m,n})$, we first describe an intermediate state $\mathbf{Y}'_n = (\gamma'_n, Y'_{1,n}, Y'_{2,n}, \ldots, Y'_{m,n})$ which is the result of applying step 1 below. Step 2 then takes $\mathbf{Y}'_n$ to $\mathbf{Y}_{n+1}$.

1. First we update γ_n and drop off particles that reach the baths. More precisely: If $\gamma_n \neq 1$ or N, we set $\gamma'_n = \gamma_n \pm 1$ with equal probability; all other random variables take the same values as before. If $\gamma_n = 1$, we set $\gamma'_n = 1$ or 2 with equal probability; additionally, if $\gamma'_n = 1$, we set $Y'_{j,n} = \mathrm{L}$ for all j such that

$Y_{j,n} = *$; everything else is unchanged. An analogous procedure is carried out for $\gamma_n = N$. (In the case $\gamma_n = \gamma'_n = 1$, we think of the vehicle as having made a trip to the left bath and returned instantaneously, unloading all the particles. That is to say, there is no particle with $Y'_{j,n} = *$ in this case.)

2. We update the locations of the particles: Suppose $\gamma'_n = i$. Let $\Gamma = \{j, Y'_{j,n} = i \text{ or } *\}$, and let r be the cardinality of Γ. We regard all r particles in Γ as indistinguishable, pick $p \in \{0, 1, \ldots, r\}$, each number getting picked with probability $\frac{1}{r+1}$, and select randomly a p-element subset Γ' of Γ. For $j \in \Gamma'$, set $Y_{j,n+1} = *$, and for $j \in \Gamma \setminus \Gamma'$, set $Y_{j,n+1} = i$. All other random variables are unchanged.

Observe that since γ_n is never = L or R, it follows that once a particle reaches L or R, it will remain in these locations forever.

This completes the description of $\mathbf{Y}_n$. The exact relation between $\mathbf{Y}_n$ and $\hat{\mathbf{X}}_n$ is explained in Sec. 3. Suffice it to notice here that while $\hat{\mathbf{X}}_n$ draws in samples of energies from the heat baths to build a steady state in the interior of the chain, the flow (of particles) in $\mathbf{Y}_n$ is in the reverse direction, from the interior of the chain to the two ends. This "duality" allows us to get our hands on the left side of (2) via Proposition 2.1 below.

Let x and $(m_{-\ell}, \ldots, m_\ell)$ be as in Lemma 2.2. We start $\mathbf{Y}_n$ by putting m_i particles at site $[xN] + i$, $-\ell \le i \le \ell$, these being the only particles in the system. Since every particle eventually comes to rest at either site L or site R, the asymptotic state of the process can be summarized by a probability $\nu = \nu_{x,\{m_i\},\gamma_0}$ on $\mathbb{N} \times \mathbb{N}$ defined as follows: For $(k_L, k_R) \in \mathbb{N} \times \mathbb{N}$, $\nu\{(k_L, k_R)\}$ is the probability of having k_L particles eventually resting in L and k_R in R. Let $\nu_{x,\{m_i\}} = \frac{1}{N} \sum_{\gamma_0=1}^{N} \nu_{x,\{m_i\},\gamma_0}$.

Proposition 2.1.

$$\int \Pi_{i=-\ell}^{\ell} \xi_{[xN]+i}^{m_i} \, d\hat{\mu} = \left(\Pi_{i=-\ell}^{\ell} m_i!\right) \int T_L^{k_L} T_R^{k_R} \, d\nu_{x,\{m_i\}}. \tag{3}$$

2.3. Asymptotic Independence

From Lemma 2.2 and Proposition 2.1, we see that to prove LTE for $\hat{\mathbf{X}}_n$, it remains to prove

Proposition 2.2. *For every $x \in (0, 1)$ and $\{m_i, -\ell \le i \le \ell\}$,*

$$\int T_L^{k_L} T_R^{k_R} \, d\nu_{x,\{m_i\}}^{(N)} \to T^m \quad \text{as } N \to \infty \tag{4}$$

where $T = (1 - x)T_L + xT_R$ and $m = \sum_{i=-\ell}^{\ell} m_i$.

The following, if true, would imply Proposition 2.2:

(i) In $\mathbf{Y}_n$, each particle individually performs an unbiased random walk.
(ii) The dynamics of distinct particles are independent.

We will show that (i) is in fact true modulo the time a particle spends waiting to get on the vehicle. (ii) is also valid most of the time; it fails only when there is more than one element in the set Γ in Sec. 2.2. Since in the limit in question, the total number of particles m is held fixed as N goes to infinity, it is an increasingly rare event for two particles to meet. Asymptotic independence is therefore expected.

3. MOMENTS VIA DUAL PROCESSES

The purpose of this section is to relate the invariant measures of $\hat{\mathbf{X}}_n$ to those of $\mathbf{Y}_n$ and to prove Proposition 2.1. First we dispose of some elementary facts in Sec. 3.1.

3.1. Invariant Measures

Proof of Proposition 1.1: To prove the invariance of μ^e, it suffices to show $\int d\mu^e Lf = 0$ for f which are continuous functions with compact support. For definiteness, we consider the component of μ^e corresponding to $\delta_{\alpha,i}$ for some $i \neq 1, N$. Notice the following:

(i) The factor $\sqrt{\eta}$ in the generator is cancelled by $\frac{1}{\sqrt{\eta}}$ in the density of μ^e.
(ii) The measure

$$\delta_{\alpha,i} \times d\eta e^{\beta\eta} \times d\xi_1 \cdots d\xi_N \Pi_{i=1}^N e^{-\beta\xi_i}$$

is preserved by the process of mixing energies. This is because if ζ_1 and ζ_2 are independent exponential random variables with the same parameter, and p is a uniform random variable independent of ζ_1 or ζ_2, then $p(\zeta_1 + \zeta_2)$ and $(1-p)(\zeta_1 + \zeta_2)$ have the same joint distribution as ζ_1 and ζ_2.
(iii) Tracer movement transforms the measure in (ii) to

$$\frac{1}{2}(\delta_{\alpha,i-1} + \delta_{\alpha,i+1}) \times d\eta e^{\beta\eta} \times d\xi_1 \cdots d\xi_N \Pi_{i=1}^N e^{-\beta\xi_i}.$$

Treating the boundary cases separately (we leave that to the reader) and summing over i, we conclude that $\int d\mu^e Lf = 0$.

Uniqueness follows from Doeblin's condition. □

Proof of Lemma 2.1: We assume, without loss of generality, that the mean bath temperatures for $\hat{\mathbf{X}}_n$ are T_L and T_R with $T_L \leq T_R$. Let $\hat{\mathbf{X}}'_n : (\alpha'_n, \eta'_n, \xi'_{1,N}, \ldots, \xi'_{N,n})$ be the process with rules identical to those in $\hat{\mathbf{X}}_n$ but with bath temperatures equal

to T_R at both ends. It follows from the proof of Proposition 1.1 that

$$\hat{\mu}' = \left(\frac{1}{N}\sum_{i=1}^{N}\delta_{\alpha',i}\right) \times \left(d\eta'\ \beta_R e^{-\beta_R \eta'}\right) \times \left(d\xi'_1 \cdots d\xi'_N\ \Pi_{i=1}^{N}\beta_R e^{-\beta_R \xi'_i}\right),$$

$$\beta_R = \frac{1}{T_R},$$

is an invariant measure for $\hat{\mathbf{X}}'_n$. We couple the processes $\hat{\mathbf{X}}_n$ and $\hat{\mathbf{X}}'_n$ together using a single Bernoulli random variable to determine the jumps of the tracer in both processes and a single uniform random variable for the mixing of energies. That is to say, for all n, we have $\alpha_n = \alpha'_n$ and $p_n = p'_n$ where p_n and p'_n are fractions in the mixing of energies. We will also couple the baths in such a way that $\xi_{R,n} = \xi'_{R,n}$ and $\xi_{L,n} < \xi'_{L,n}$ for all n. It follows that if we start these two processes with the same initial data, then we have $(\alpha_n, \eta_n, \xi_{1,N}, \ldots, \xi_{N,n}) \leq (\alpha'_n, \eta'_n, \xi'_{1,N}, \ldots, \xi'_{N,n})$ for all n, this inequality holding *a.s.* sample path by sample path, coordinatewise.

An invariant probability measure for $\hat{\mathbf{X}}_n$ is constructed as follows: Start $\hat{\mathbf{X}}_n$ with initial distribution $\lambda_0 = \hat{\mu}'$, and let λ_n denote the distribution of $\hat{\mathbf{X}}_n$ at time n. Since $\frac{1}{n}\sum_{k=0}^{n-1}\lambda_k \leq \hat{\mu}'$ [4] for all n, tightness is guaranteed and a subsequence of $\frac{1}{n}\sum_{k=0}^{n-1}\lambda_k$ converges to an invariant probability $\hat{\mu}$ for $\hat{\mathbf{X}}_n$.

Mixing and convergence to stationary distribution from arbitrary initial data follow from standard arguments. □

Proof of Lemma 2.2: Continuing to assume $T_L \leq T_R$, we deduce from the proof of Lemma 2.1 that for each N, $\hat{\mu}^{(N)}$ is bounded above by a product of exponentials of rate $\beta_R = \frac{1}{T_R}$. The same, therefore, is true for the projected measures $\hat{\mu}^{(N)}_{x,\ell}$. Since the sequence $\{\hat{\mu}^{(N)}_{x,\ell},\ N > 2\ell + 1\}$ is tight, a subsequence converges to some $\hat{\mu}_{x,\ell}$. It then follows from (2) that

$$E_{\hat{\mu}_{x,\ell}}\left[\Pi_{i=-\ell}^{\ell}\ \xi_{[xN]+i}^{m_i}\right] \ =\ \left(\Pi_{i=-\ell}^{\ell} m_i!\right)\beta^{-(m_{-\ell}+\cdots+m_\ell)}. \tag{5}$$

This is true for every $(2\ell+1)$-tuple of nonnegative integers $(m_{-\ell}, \ldots, m_\ell)$. Since probability measures bounded above by a fixed product of exponentials are uniquely identified by their moments, (5) forces

$$\hat{\mu}_{x,\ell} \ =\ d\zeta_{-\ell}\cdots d\zeta_\ell\ \Pi_{i=-\ell}^{\ell}\beta e^{-\beta\zeta_i}.$$

Convergence follows from the fact that the argument above is valid for every accumulation point of $\{\hat{\mu}^{(N)}_{x,\ell},\ N > 2\ell + 1\}$. □

[4] $\nu \leq \mu$ means as usual that $\int f d\nu \leq \int f d\mu$ for all increasing functions f.

3.2. Duality: Precise Formulation

To be technically correct, the process that is dual to $\hat{\mathbf{X}}_n$ is not $\mathbf{Y}_n$ but another process $\kappa(\mathbf{Y}_n)$ that keeps count of the number of particles in the various locations for $\mathbf{Y}_n$, and this duality is valid only with respect to a certain function. We now make precise these assertions.

The process $\kappa(\mathbf{Y}_n)$ is defined by random variables

$$(\gamma_n, q_n;\ k_{1,n}, \ldots, k_{N,n};\ k_{L,n}, k_{R,n}), \quad n = 0, 1, 2, \ldots.$$

We first define its evolution in terms of that of $\mathbf{Y}_n$: γ_n is as in $\mathbf{Y}_n$; q_n is the number of particles "in transit", i.e., the number of j such that $Y_{j,n} = *$; $k_{i,n}$ is the number of j such that $Y_{j,n} = i$, and $k_{L,n}$ (resp. $k_{R,n}$) the number of j such that $Y_{j,n} = L$ (resp. R). Having defined $\kappa(\mathbf{Y}_n)$ this way, i.e., by ignoring the labels on the particles in $\mathbf{Y}_n$, we observe that it is a *bona fide* Markov chain. This is because if we permute the names of the particles at step n, the particle count at step $n + 1$ is not affected.

Clearly, starting with any initial condition, $q_n = k_{i,n} = 0$ for all i as $n \to \infty$. Let ν be the probability measure on $\mathbb{N} \times \mathbb{N}$ introduced immediately before Proposition 2.1 and observe that ν is in fact the asymptotic distribution for $\kappa(\mathbf{Y}_n)$.

Let $\underline{\xi} = (\eta, \xi_1, \ldots, \xi_N)$ and $\underline{k} = (q, k_1, \ldots, k_N, k_L, k_R)$ be shorthand for these variables. The function with respect to which duality between $\hat{\mathbf{X}}_n$ and $\kappa(\mathbf{Y}_n)$ is asserted is

$$F(\underline{\xi}, \underline{k}) = \left(\Pi_{i=1}^{N} \frac{\xi_i^{k_i}}{k_i!}\right) \frac{\eta^q}{q!}\, T_L^{k_L} T_R^{k_R}$$

where $\beta_L^{-1} = T_L$ and $\beta_R^{-1} = T_R$. The next proposition is a statement of this duality for individual tracer paths:

Proposition 3.1. *Let $\underline{\xi}^*$ and $\underline{k}^*$ be fixed values of $\underline{\xi}$ and $\underline{k}$, and let $(i_0, i_1, \ldots, i_n)$ be an admissible tracer path. Then*

$$\begin{aligned} &E[F(\underline{\xi}_n, \underline{k}^*) | \underline{\xi}_0 = \underline{\xi}^*, \alpha_0 = i_0, \ldots, \alpha_n = i_n] \\ &\quad = E[F(\underline{\xi}^*, \underline{k}_n) | \underline{k}_0 = \underline{k}^*, \gamma_0 = i_n, \ldots, \gamma_n = i_0]. \end{aligned}$$

Proposition 3.1 is proved in the next subsection. Observe that γ runs in the reverse direction as α. Averaging over all tracer paths (all of which are weighted equally), we obtain

$$E[F(\underline{\xi}_n, \underline{k}^*) \mid \underline{\xi}_0 = \underline{\xi}^*] = E[F(\underline{\xi}^*, \underline{k}_n) \mid \underline{k}_0 = \underline{k}^*]. \tag{6}$$

Equation (6) holds for every n. Letting $n \to \infty$, we obtain

Corollary 2. *For every* $\underline{k}^*$,

$$\int F(\cdot, \underline{k}^*)\, d\hat{\mu} \;=\; \int T_L^{k_L} T_R^{k_R}\; d\nu_{\underline{k}^*}$$

where $\nu_{\underline{k}^*}$ *is the probability on* $\mathbb{N} \times \mathbb{N}$ *describing the asymptotic distribution of particles starting from the configuration* $\underline{k}^*$.

Proof: Fix some arbitrary $\underline{\xi}^*$. By Lemma 2.1, the distribution of $(\alpha_n, \underline{\xi}_n)$ converges to $\hat{\mu}$ as $n \to \infty$. Thus the expectation on the left side of (6) converges to $\int F(\underline{\xi}, \underline{k}^*)\, d\hat{\mu}$. Also, the expectation on the right converges to $\int F(\underline{\xi}^*, \underline{k})\, d\nu_{\underline{k}^*}$, which does not depend on $\underline{\xi}^*$ and is equal to $\int T_L^{k_L} T_R^{k_R} d\nu_{\underline{k}^*}$. □

Proposition 2.1 follows from Corollary 2 by setting all the coordinates in $\underline{k}^*$ to 0 except for $k_{[xN]-\ell}, \ldots, k_{[xN]+\ell}$, which we set equal to $m_{-\ell}, \ldots, m_\ell$.

3.3. Proof of Duality

We write $\underline{\xi}^* = (\eta^*, \xi_1^*, \ldots, \xi_N^*)$ and $\underline{k}^* = (q^*, k_1^*, \ldots, k_N^*, k_L^*, k_R^*)$.

Lemma 3.1. *Given* $(\underline{\xi}^*, \underline{k}^*)$ *and* (i_0, i_1)*, we have*

$$\begin{aligned} &E[F(\underline{\xi}_1, \underline{k}^*) | \underline{\xi}_0 = \underline{\xi}^*, \alpha_0 = i_0, \alpha_1 = i_1] \\ &\quad = E[F(\underline{\xi}^*, \underline{k}_1) | \underline{k}_0 = \underline{k}^*, \gamma_0 = i_1, \gamma_1 = i_0]. \end{aligned} \tag{7}$$

Proof:
Case 1. $(i_0, i_1) \neq (1, 1)$ or (N, N), i.e., no baths are involved. We write the left side of (7) as $I \cdot II$ where

$$I = \left(\Pi_{i \neq i_0} \frac{\xi_i^{*k_i^*}}{k_i^*!} \right) T_L^{k_L^*} T_R^{k_R^*}$$

and

$$II \;=\; E\left[\frac{\xi_{i_0,1}^{k_{i_0}^*}}{k_{i_0}^*!} \frac{\eta_1^{q^*}}{q^*!} \mid \underline{\xi}_0 = \underline{\xi}^*, \alpha_0 = i_0, \alpha_1 = i_1 \right].$$

Writing $k = k_{i_0}^*$, $q = q^*$, $\xi = \xi_{i_0}^*$ and $\eta = \eta^*$ to simplify notation, we obtain

$$\begin{aligned} II &= \frac{1}{k!q!} \int_0^1 dp\, [p(\xi + \eta)]^k\, [(1-p)(\xi + \eta)]^q \\ &= \frac{1}{k!q!} (\xi + \eta)^{k+q} \int_0^1 dp\, (p^k (1-p)^q) \end{aligned}$$

$$= \frac{1}{k!q!} \left\{ \sum_{m=0}^{k+q} \frac{(k+q)!}{m!(k+q-m)!} \xi^m \eta^{k+q-m} \right\} \frac{k!q!}{(k+q)!(k+q+1)}$$

$$= \frac{1}{k+q+1} \sum_{m=0}^{k+q} \frac{\xi^m}{m!} \frac{\eta^{k+q-m}}{(k+q-m)!}.$$

With II in this form, we see immediately that the right side of (7) is also equal to $I \cdot II$. Notice that by design, mixing takes place at site i_0 in both processes.

Case 2. $i_0 = i_1 = 1$ or N. We treat the case $i_0 = i_1 = 1$. Here again we write the left side of (7) as $I \cdot II$ where I is as in Case 1 and

$$II = \frac{1}{k!} \int_0^1 dp\, [p(\xi+\eta)]^k \cdot \frac{1}{q!} \int_0^\infty d\sigma\, \sigma^q \beta_L e^{-\beta_L \sigma}$$

$$= \frac{1}{k+1} \sum_{m=0}^{k} \frac{\xi^m}{m!} \frac{\eta^{k-m}}{(k-m)!} \cdot \beta_L^{-q}.$$

The first part of II is obtained by setting $q = 0$ in the computation of II in Case 1; the second part is a standard fact. Notice that setting $q_1 = 0$ and $k_{L,1} = k_L^* + q$ is exactly what is done to compute the right side of (7). □

Proof of Proposition 3.1: Inducting on n, we have, for $n \geq 2$,

$$E[F(\underline{\xi}_n, \underline{k}^*) | \underline{\xi}_0 = \underline{\xi}^*, \alpha_0 = i_0, \ldots, \alpha_n = i_n]$$

$$= \int E[F(\underline{\xi}_n, \underline{k}^*) | \underline{\xi}_1 = \underline{\xi}', i_1, \ldots, i_n] \cdot P(\underline{\xi}_1 = \underline{\xi}' | \underline{\xi}_0 = \underline{\xi}^*, i_0, i_1)\, d\underline{\xi}'$$

$$= \int E[F(\underline{\xi}', \underline{k}_{n-1}) | \underline{k}_0 = \underline{k}^*, i_n, \ldots, i_1] \cdot P(\underline{\xi}_1 = \underline{\xi}' | \underline{\xi}_0 = \underline{\xi}^*, i_0, i_1)\, d\underline{\xi}'$$

$$= \int F(\underline{\xi}', \underline{k}') \cdot P(\underline{k}_{n-1} = \underline{k}' | \underline{k}_0 = \underline{k}^*, i_n, \ldots, i_1) \cdot P(\underline{\xi}_1 = \underline{\xi}' | \underline{\xi}_0 = \underline{\xi}^*, i_0, i_1)\, d\underline{\xi}'\, d\underline{k}'$$

$$= \int E[F(\underline{\xi}_1, \underline{k}') | \underline{\xi}_0 = \underline{\xi}^*, i_0, i_1] \cdot P(\underline{k}_{n-1} = \underline{k}' | \underline{k}_0 = \underline{k}^*, i_n, \ldots, i_1)\, d\underline{k}'$$

$$= \int E[F(\underline{\xi}^*, \underline{k}_n) | \underline{k}_{n-1} = \underline{k}', i_1, i_0] \cdot P(\underline{k}_{n-1} = \underline{k}' | \underline{k}_0 = \underline{k}^*, i_n, \ldots, i_1)\, d\underline{k}'$$

$$= E[F(\underline{\xi}^*, \underline{k}_n) | \underline{k}_0 = \underline{k}^*, \gamma_0 = i_n, \ldots, \gamma_n = i_0].$$

The induction hypothesis is invoked at the second equal sign on the stretch of tracer path $(i_1, \ldots, i_n)$, and again at the second to last equal sign for (i_0, i_1). □

4. ASYMPTOTIC INDEPENDENCE IN PARTICLE SYSTEMS

The purpose of this section is to prove Proposition 2.2, thereby completing the proof of LTE for $\hat{\mathbf{X}}_n$.

4.1. A Slight Simplification

We introduce here a modification of $\mathbf{Y}_n$. The resulting process, which we call $\tilde{\mathbf{Y}}_n$, is a little simpler to work with. Given $\tilde{\mathbf{Y}}_n = (\gamma_n, Y_{1,n}, Y_{2,n}, \ldots, Y_{m,n})$, we obtain $\tilde{\mathbf{Y}}_{n+1}$ by first performing Step 2 in the definition of $\mathbf{Y}_n$ before Step 1 (see Sec. 2.2). That is to say, if $\gamma_n = i$, then we first select from among those particles j with $Y_{j,n} = i$ or $Y_{j,n} = *$ which ones will board the vehicle *before* the vehicle takes off. With this order of updating, there is no need to distinguish between $Y_{j,n} = i$ and $Y_{j,n} = *$ when $\gamma_n = i$. Thus we combine the two, setting $Y_{j,n} = i$ for those j with $Y_{j,n} = *$. Using this convention, $Y_{j,n}$ takes values only in $\{1, \ldots, N\} \cup \{L, R\}$. In the rest of this section, we will also use L and R synonymously with 0 and $N + 1$.

What is needed in Proposition 2.2 is the asymptotic distribution of $\mathbf{Y}_n$ given $\mathbf{Y}_0 = (\gamma_0, Y_{1,0}, Y_{2,0}, \ldots, Y_{m,0})$. After carrying out Step 1 in the definition of $\mathbf{Y}_n$, we view the resulting Bernoulli measure as initial distribution for the process $\tilde{\mathbf{Y}}_n$. The problem is thus reduced to the asymptotic distributions of $\tilde{\mathbf{Y}}_n$. (Notice that in the starting configurations of interest, there is, in fact, no j with $Y_{j,0} = *$; see Sec. 3.2. Thus the two states in this Bernoulli measure differ from $\mathbf{Y}_0$ only in γ_0.)

From here on we will work exclusively with $\tilde{\mathbf{Y}}_n$.

4.2. Preliminary Observations

We introduce the notation $Y_{j,\infty} = L$ or R if $\lim_{n\to\infty} Y_{j,n} = L$ or R.

Lemma 4.1. *Suppose there is a single particle in the system. Then modulo waiting time (i.e., the times when the particle does not move), the motion of the particle is that of a symmetric random walk. Consequently, if $Y_{1,0} = i$, then*

$$P(Y_{1,\infty} = L) = \frac{N - i + 1}{N + 1}, \quad P(Y_{1,\infty} = R) = \frac{i}{N + 1}.$$

This is quite obvious.

Lemma 4.2. *The dynamics of a system (with a fixed number of particles) are unaffected by the addition of new particles.*

Proof: Consider a system with m particles. For definiteness, assume $\gamma_n = i$, and $Y_{j,n} = i$ for $1 \le j \le k$, $Y_{j,n} \ne i$ for $k < j \le m$. Let $J(j_1, \ldots, j_\ell)$ denote the probability that particles $j_1, \ldots, j_\ell$ jump at this time and the remaining $m - \ell$ particles do not. Then

$$J(j_1, \ldots, j_\ell) = \frac{1}{k+1} \cdot \frac{1}{C_\ell^k} = \frac{\ell!(k-\ell)!}{(k+1)!}.$$

where C_ℓ^k is the number of ways of choosing ℓ particles from k particles. Now add a new particle to the system. Assume $Y_{m+1,n} = i$, and the situation with respect to the original m particles is as before. Then

$$J(j_1, \ldots, j_\ell) + J(j_1, \ldots, j_\ell, m+1) = \frac{1}{k+2}\frac{1}{C_\ell^{k+1}} + \frac{1}{k+2}\frac{1}{C_{\ell+1}^{k+1}} = \frac{\ell!(k-\ell)!}{(k+1)!},$$

proving that at this step, the movements of the m particles in the original system are oblivious to the presence of the $(m+1)$st particle. □

Corollary 3. *Modulo waiting times, every particle individually performs a symmetric random walk until it reaches L or R.*

This follows immediately from Lemmas 4.1 and 4.2. What we need to establish is the *independence* of these walks.

Setting and notation for the rest of this section:

Results for the following two kinds of initial configurations are needed:

(1) As in Sec. 2.2, we let $x \in (0, 1)$ and $m_{-\ell}, \ldots, m_\ell$ be fixed. For each N, we start with m_i particles at site $[xN] + i$, $-\ell \le i \le \ell$, and no other particles in the system.
(2) In addition to the m_i particles at sites $[xN] + i$, $-\ell \le i \le \ell$, we put $\bar{m}$ particles at site $[yN]$ for $y \ne x$. These are the only particles in the system.

In each of the two cases, we run $\tilde{\mathbf{Y}}_n$, and obtain for each γ_0 an asymptotic distribution. We estimate the distance between this distributions and the product measure corresponding independent walks, taking supremum as γ_0 ranges over all initial positions. Finally, we show that these maximum errors go to 0 as $N \to \infty$.

With obvious modifications, our arguments are valid when x in Case (1) above is replaced by a finite number of points $x_1, \ldots, x_r \in (0, 1)$, each with its own set of particles in a block centered at $[x_i N]$. The arguments are essentially unchanged. For definiteness and to avoid cumbersome notation, we treat only Case (1).

Let $m = \sum_{i=-\ell}^{\ell} m_i$. For an m-block $(A_1, \ldots, A_m)$ where each $A_j = L$ or R, let

$$P(A_1, \ldots, A_m) = P(Y_{1,\infty} = A_1, \ldots, Y_{m,\infty} = A_m).$$

When it is necessary to emphasize the number of entries in the block, we will write $P_m(\cdots)$. In view of Corollary 3, asymptotic independence means for all $(A_1, \ldots, A_m)$,

$$P(A_1, \ldots, A_m) \to Q_{A_1}(x) \cdots Q_{A_m}(x) \quad \text{as } N \to \infty \quad \text{uniformly in } \gamma_0$$

where $Q_L(x) = 1 - x$, $Q_R(x) = x$.

Lemma 4.3. *It suffices to prove that for all $m \geq 2$,*

$$P_m(R, R, \ldots, R) \to x^m \quad \text{as } N \to \infty.$$

Proof: We show how to deduce $P_m(A_1, \ldots, A_m)$ for all $(A_1, \ldots, A_m)$ from the number $P_m(R, R, \ldots, R)$ together with knowledge of $P_k(A_1, \ldots, A_k)$ for all $(A_1, \ldots, A_k)$, $k < m$. Since $P_m(L \text{ or } R, R, \ldots, R) = P_{m-1}(R, \ldots, R)$ (Lemma 4.2), it follows from

$$P_m(L, R, \ldots, R) = P_m(L \text{ or } R, R, \ldots, R) - P_m(R, R, \ldots, R)$$

that $P_m(L, R, \ldots, R)$ is determined. Likewise,

$$P_m(L, L, R, \ldots, R) = P_m(L, L \text{ or } R, R, \ldots, R) - P_m(L, R, R, \ldots, R),$$

and so on. □

4.3. Lower Bound of $P(R, R, \ldots, R)$

Proposition 4.1. $\liminf_{N\to\infty} P(R, R, \ldots, R) \geq x^m$.

This is the easier of the two inequalities because compared to independent random walks, the particles in our system have a tendency to stick together when they meet. To illustrate this point, consider the case of 2 particles, with $Y_{1,n} = Y_{2,n} = \gamma_n = i$. Then $P(Y_{1,n+1} = Y_{2,n+1}) = \frac{2}{3}$, compared to $\frac{1}{2}$ in two independent walks.

Let $\mathcal{F}_n$ be the σ-algebra generated by the random variables $\{Y_{j,r}, \gamma_r, 1 \leq j \leq m, 0 \leq r \leq n\}$. Let

$$f(Y_1, \ldots, Y_m) := Y_1 \cdots Y_m,$$

and define

$$\Delta f(Y_{1,n}, \ldots, Y_{m,n}; \gamma_n)$$
$$:= E[f(Y_{1,n+1}, \ldots, Y_{m,n+1}) | Y_{i,n}, \ldots, Y_{m,n}; \gamma_n] - f(Y_{1,n}, \ldots, Y_{m,n}).$$

Since this definition is independent of n, the notation $\Delta f(Y, \ldots, Y_m; \gamma)$ makes sense.

Lemma 4.4. *The process* $f(Y_{1,n}, \ldots, Y_{m,n})$, $n = 0, 1, \ldots$, *is a submartingale with respect to the filtration* $\{\mathcal{F}_n\}$.

Proof: If suffices to show

$$\Delta f(Y_1, \ldots, Y_m; i) \geq 0 \quad \text{for all}(Y_1, \ldots, Y_m; i).$$

Fix $Y_1, \ldots, Y_m$ and i. Clearly, if $Y_j = 0$ for some j, then $\Delta f = 0$, so we may assume $Y_j \neq 0$ for all j. Since Δf depends only on changes in location of those particles with $Y_j = i$, it suffices to show $\Delta \hat{f} \geq 0$ where $\Delta \hat{f}$ is defined by

$$\Delta f = \left(\Pi_{Y_j \neq i} Y_j\right) \cdot \Delta \hat{f}.$$

Let k be the number of j with $Y_j = i$, and let J be the number of particles that jump. Clearly, $E[\Delta \hat{f} | J = 0] = 0$. For $p = 1, \ldots, k$,

$$E[\Delta \hat{f} | J = p] = i^{k-p} \cdot \left\{ \frac{1}{2}((i-1)^p + (i+1)^p) - i^p \right\}. \tag{8}$$

This is $= 0$ if $p = 1$, > 0 by Jensen's inequality if $p > 1$. □

The notation $O(N^r)$ is used to denote a term that is less than CN^r with C independent of N. We record for future use the following observation:

Lemma 4.5. *Given* $(Y_1, \ldots, Y_m; i)$, *let* k *be the number of* j *with* $Y_j = i$. *Then* $\Delta f(Y_1, \ldots, Y_m; i) = 0$ *unless both (i)* $Y_j \neq 0$ *for all* j, *and (ii)* $k \geq 2$ *hold. When both (i) and (ii) hold, we have*

$$\Delta f(Y_1, \ldots, Y_m; i) = c(k) \cdot (\Pi_{1 \leq j \leq m} Y_j) i^{-2} + O(N^{m-3}) \tag{9}$$

where $c(k)$ *is a constant depending only on* k.

Proof: That $\Delta f = 0$ unless both (i) and (ii) hold is proved in Lemma 4.4. If (i) and (ii) both hold, then it follows from (8) that

$$\begin{aligned} \Delta f(Y_1, \ldots, Y_m; i) &= \frac{1}{k+1} \sum_{p=0}^{k} E[\Delta f | J = p] \\ &= \frac{1}{k+1} \sum_{p=2}^{k} C_p^k \cdot \left(\Pi_{Y_j \neq i} Y_j\right) \cdot i^{k-p} \cdot C_2^p i^{p-2} + O(N^{m-3}), \end{aligned}$$

which has the form of the expression on the right side of (9). □

Proof of Proposition 4.1: Since $f(Y_{1,n}, \ldots, Y_{m,n})$ is a submartingale, it converges almost surely to $f(Y_{1,\infty}, \ldots, Y_{m,\infty})$, which is $= 0$ except when $Y_{j,\infty} = R$ for all

j, where it is $= (N+1)^m$. Now

$$f(Y_{1,0}, \ldots, Y_{m,0}) \approx [xN]^m,$$

and inductively, we have, for all $n \geq 0$,

$$E[f(Y_{1,n}, \ldots, Y_{m,n})] - f(Y_{1,0}, \ldots, Y_{m,0}) = \sum_{r=0}^{n-1} E[\Delta f(Y_{1,r}, \ldots, Y_{m,r}; \gamma_r)]. \tag{10}$$

By Lemma 4.4, the random variables $\Delta f(Y_{1,r}, \ldots, Y_{m,r}; \gamma_r)$ are ≥ 0 everywhere. Hence $E[f(Y_{1,n}, \ldots, Y_{m,n})] \geq f(Y_{1,0}, \ldots, Y_{m,0})$. It follows that

$$P(R, R, \ldots, R) \;=\; \frac{1}{(N+1)^m} E[f(Y_{1,\infty}, \ldots, Y_{m,\infty})] \;\geq\; \frac{1}{(N+1)^m} [xN]^m,$$

which tends to x^m as $N \to \infty$. □

4.4. Upper Bound of $P(R, R, \ldots, R)$

Proposition 4.2. $\limsup_{N\to\infty} P(R, R, \ldots, R) \leq x^m$.

To prove this inequality, we consider $f = f_1 - af_2$ where $f_1(Y_1, \ldots, Y_m) = \Pi_j Y_j$,

$$f_2 = \sum_{1 \leq j_1 < j_2 \leq m} g_{j_1 j_2}, \quad g_{j_1 j_2}(Y_1, \ldots, Y_m) = |Y_{j_1} - Y_{j_2}| \cdot \left(\Pi_{j \neq j_1, j_2} Y_j\right),$$

and $a > 0$ is a constant to be determined. We consider separately $\Delta f_1(Y_1, \ldots, Y_m; i)$ and $\Delta f_2(Y_1, \ldots, Y_m; i)$, in each case distinguishing between terms that are of order N^{m-2} and $O(N^{m-3})$. Observe that no terms of order N^{m-1} or higher are present: we have shown this for Δf_1; for Δf_2, this is obvious since $f_2 = O(N^{m-1})$.

Lemma 4.6. *To show* $\limsup_{N\to\infty} P(R, R, \ldots, R) \leq x_1 \cdots x_m$, *it suffices to show that the* N^{m-2}*-terms in* $\Delta f(Y_1, \ldots, Y_m; i)$ *are* ≤ 0 *for all* $(Y_1, \ldots, Y_m; i)$. *That is to say, the* $O(N^{m-3})$*-terms play no role.*

Proof: Given $(Y_{1,0}, \ldots, Y_{m,0}; \gamma_0)$, we introduce a sequence of stopping times as follows: S is the smallest n such that $Y_{j,n} = L$ or R for all j, and $S_0 = 0$. Let $0 < \varepsilon < 1$ be fixed. After S_q is defined, we let

$$S_{q+1} := \min\{S,\ \inf\{n > S_q : Y_{j,n} \neq Y_{j,n-1} \text{ for some } j\}\}.$$

Let $S^* = S_{mN^{2+\varepsilon}}$. Then if $S > S^*$, at least one of the particles jumps $> N^{2+\varepsilon}$ times without reaching L or R. Recall that ignoring waiting time, the motion of

each particle individually is that of an unbiased random walk (Corollary 3). The probability of such a particle not reaching L or R in $\geq N^{2+\varepsilon}$ steps is $< cN^{-\frac{\varepsilon}{2}}$ for large N. Thus $P(S > S^*) < cmN^{-\frac{\varepsilon}{2}} := \varepsilon_N$, and

$$\begin{aligned} E[f_1(Y_{1,S^*}, \ldots, Y_{m,S^*})] &\geq E[f_1(Y_{1,S^*}, \ldots, Y_{m,S^*})|S^* = S] \cdot P(S = S^*) \\ &\geq E[f_1(Y_{1,\infty}, \ldots, Y_{m,\infty})|S^* = S] \cdot P(S = S^*) \\ &\geq E[f_1(Y_{1,\infty}, \ldots, Y_{m,\infty})] - (N+1)^m P(S > S^*) \\ &\geq (P(\mathrm{R}, \ldots, \mathrm{R}) - \varepsilon_N) \cdot (N+1)^m. \end{aligned}$$

Since $E[f_2(Y_{1,S^*}, \ldots, Y_{m,S^*})] = O(N^{m-1})$, it follows from the inequalities above that

$$(P(R, \ldots, R) - \varepsilon_N)(N+1)^m \leq E[f(Y_{1,S^*}, \ldots, Y_{m,S^*})] \; + \; O(N^{m-1}).$$

We estimate $E[f(Y_{1,S^*}, \ldots, Y_{m,S^*})]$ as follows. Let $\Omega_{q,n}$ denote the set of sample paths $\omega^n = \{(Y_{i,r}, \ldots, Y_{m,r}; \gamma_r), r \leq n\}$ such that $S_q(\omega^n) = n$ and $S(\omega^n) \geq n$. Then

$$\begin{aligned} &E[f(Y_{1,S^*}, \ldots, Y_{m,S^*})] \; - \; [xN]^m \\ &\approx \sum_{q=1}^{mN^{2+\varepsilon}} \left\{ \sum_{n=1}^{\infty} \sum_{\omega^n \in \Omega_{q,n}} (f(Y_{1,n} \ldots, Y_{m,n}) - f(Y_{1,n-1}, \ldots, Y_{m,n-1}))P(\omega^n) \right\}. \end{aligned} \tag{11}$$

Now fix $\hat{\omega}^{n-1} = \{(\hat{Y}_{i,r}, \ldots, \hat{Y}_{m,r}; \hat{\gamma}_r), r \leq n-1\}$, and let $\Omega_{q,n}(\hat{\omega}^{n-1})$ be the subset of $\Omega_{q,n}$ consisting of paths which up to time $n-1$ coincide with $\hat{\omega}^{n-1}$. Then the part of the sum over $\omega^n \in \Omega_{q,n}$ corresponding to $\omega^n \in \Omega_{q,n}(\hat{\omega}^{n-1})$ is

$$\left[\sum_{\omega^n \in \Omega_{q,n}(\hat{\omega}^{n-1})} (f(Y_{1,n}, \ldots, Y_{m,n}) - f(\hat{Y}_{1,n-1}, \ldots, \hat{Y}_{m,n-1})) \cdot \frac{P(\omega^n)}{\sum P(\omega^n)} \right] \cdot \sum P(\omega^n).$$

The quantity between the square brackets

$$= \; \frac{k+1}{k} \cdot \Delta f(\hat{Y}_{1,n-1}, \ldots, \hat{Y}_{m,n-1}; \hat{\gamma}_{n-1}) \tag{12}$$

where k is the number of j for which $\hat{Y}_{j,n-1} = \hat{\gamma}_{n-1}$. The $\frac{k+1}{k}$ correction reflects the fact that with probability $\frac{1}{k+1}$, no particle jumps.

We have shown that the quantities inside curly brackets in (11) are weighted averages of terms of the form in (12). Write

$$E[f(Y_{1,S^*}, \ldots, Y_{m,S^*})] \; = \; [xN]^m \; + \; \Delta^{(2)} \; + \; \Delta^{(3)}$$

where $\Delta^{(2)}$ and $\Delta^{(3)}$ are the N^{m-2} and $O(N^{m-3})$-contributions respectively. Since there are at most $mN^{2+\epsilon}$ terms of the type inside curly brackets, it follows that

$$\Delta^{(3)} \;=\; O(N^{m-3+2+\varepsilon}) \;=\; O(N^{m-1+\varepsilon})\,.$$

Altogether, these inequalities imply

$$P(R,\ldots,R) - \varepsilon_N \;\le\; \frac{1}{N^m}[xN]^m \;+\; \frac{1}{N^m}\Delta^{(2)} \;+\; O(N^{\varepsilon-1})\,,$$

proving the desired inequality as $N \to \infty$ if $\Delta^{(2)} \le 0$. □

The rest of the proof is concerned with the sign of $\Delta f(Y_1,\ldots,Y_m;i)$.

Lemma 4.7. *Let $(Y_1,\ldots,Y_m;i)$ be such that $Y_j = i$ for all $j \le k$ and $Y_j \ne i$ for $j > k$. Let $j_1 < j_2$ be arbitrary, and let $g = g_{j_1 j_2}$. For $1 \le p \le k$, let*

$$\Delta \;=\; E[\Delta g(Y_1,\ldots,Y_m;i) \mid \text{particles } 1,\ldots,p \text{ jump; the rest do not}].$$

Then $\Delta = O(N^{m-3})$ except when $Y_{j_1} = Y_{j_2} = i$ and $j_1 \le p < j_2$, in which case

$$\Delta \;=\; \left(\Pi_{j=1}^m Y_j\right) i^{-2} \;+\; O(N^{m-3})\,.$$

Proof: To simplify notation, we write $Z = \Pi_{j>p,\, j\ne j_1,j_2} Y_j$ and consider the following 3 scenarios:

Scenario 1. Particles j_1 and j_2 both jump. Here $g = 0$ because $|Y_{j_1} - Y_{j_2}| = 0$. Since all particles that jump end up in the same location, after the jump we again have $|(Y_{j_1} \pm 1) - (Y_{j_2} \pm 1)| = 0$. Thus $\Delta = 0$.

Scenario 2. Neither j_1 nor j_2 jumps. If $Y_{j_1} = Y_{j_2}$, then $\Delta = 0$. If $Y_{j_1} \ne Y_{j_2}$, then $g = |Y_{j_1} - Y_{j_2}| \cdot Z \cdot i^p$. After the jump, the expected value is $|Y_{j_1} - Y_{j_2}| \cdot Z \cdot \frac{1}{2}((i+1)^p + (i-1)^p)$. An argument similar to that in the previous subsection gives $\Delta = O(N^{m-3})$.

Scenario 3. j_1 jumps but not j_2. In this case, $Z = O(N^{m-p-1})$. We treat separately the following two subcases.

Case 1. $|Y_{j_1} - Y_{j_2}| \ge 1$. If $p = 1$, then $\Delta = 0$ (easy). Suppose $p > 1$.

Configuration 1: Choosing + causes $|Y_{j_1} - Y_{j_2}|$ to increase by 1. Here $g = |Y_{j_1} - Y_{j_2}| \cdot Z \cdot i^{p-1}$. After the jump, the expected value is

$$\frac{1}{2}\{(|Y_{j_1} - Y_{j_2}| + 1) \cdot Z \cdot (i+1)^{p-1} + (|Y_{j_1} - Y_{j_2}| - 1) \cdot Z \cdot (i-1)^{p-1}\}$$

This gives $\Delta = I + II$ where

$$I = |Y_{j_1} - Y_{j_2}| \cdot Z \cdot \left\{ \frac{1}{2}((i+1)^{p-1} + (i-1)^{p-1}) - i^{p-1} \right\}$$

and

$$II = Z \cdot \frac{1}{2}\{(i+1)^{p-1} - (i-1)^{p-1}\}.$$

Both I and II are $O(N^{m-3})$.

Configuration 2: Choosing + causes $|Y_{j_1} - Y_{j_2}|$ to decrease by 1. A similar computation gives $\Delta = I - II$ where I and II are as in Configuration 1.

Case 2. $|Y_{j_1} - Y_{j_2}| = 0$. Here $g = 0$. After the jump, $|Y_{j_1} - Y_{j_2}| = 1$ and the expected value is $Z \cdot \frac{1}{2}((i+1)^{p-1} + (i-1)^{p-1})$, which is $= Z \cdot i^{p-1} + O(N^{m-3})$. Now the present case (Scenario 3, Case 2) corresponds to $Y_{j_1} = Y_{j_2} = i$ and $j_1 \leq p < j_2$. In this case,

$$Z \cdot i^{p-1} + O(N^{m-3}) = \left(\Pi_{j=1}^{m} Y_j\right) i^{-2} + O(N^{m-3}).$$

□

Corollary 4. *Let $(Y_1, \ldots, Y_m; i)$ be such that the number of j with $Y_j = i$ is equal to k. Then there exists $j_1 < j_2$ such that*

$$\Delta g_{j_1, j_2}(Y_1, \ldots, Y_m; i) \geq c'(k) \cdot \left(\Pi_{j=1}^{m} Y_j\right) i^{-2} + O(N^{m-3})$$

where $c'(k) \geq \frac{1}{k(1+k)}$.

Proof: By relabeling the names of the particles, we may assume $Y_j = i$ for $j \leq k$. Choose $j_2 = k$, $j_1 < j_2$, and let $p = k - 1$. Then the probability of having exactly the first p particles jump is $\frac{1}{k(k+1)}$, and the lower bound follows from Lemma 4.7. □

Let us use "$A \geq B \bmod O(N^r)$" as shorthand for the obvious.

Proof of Proposition 4.2: We will show that for a sufficiently large,

$$a\Delta f_2(Y_1, \ldots, Y_m; i) \geq \Delta f_1(Y_1, \ldots, Y_m; i) \mod O(N^{m-3}) \tag{13}$$

for every $(Y_1, \ldots, Y_m; i)$. Proposition 4.2 then follows from Lemma 4.6.

To prove (13), observe first that $\Delta f_2 \geq 0 \bmod O(N^{m-3})$ because it is a sum of terms of the type in Lemma 4.7. Thus it suffices to consider those $(Y_1, \ldots, Y_m; i)$ satisfying conditions (i) and (ii) in Lemma 4.5. For each such $(Y_1, \ldots, Y_m; i)$, we need the N^{m-2}-term of $a\Delta f_2(Y_1, \ldots, Y_m; i)$ to compensate the corresponding (strictly positive) term in Δf_1, namely $c(k)(\Pi_{j=1^m} Y_j)i^{-2}$. Corollary 4 guarantees

a term of of exactly the same form. Notice that $\Pi_{j=1}^{m} Y_j > 0$ by condition (i). It suffices, therefore, to take $a = \max_{2\le k\le m} \frac{c(k)}{c'(k)}$. □

This completes the proof of asymptotic independence for $\tilde{\mathbf{Y}}_n$, a fact which combined with Proposition 2.1 and Lemma 2.2 gives LTE for $\hat{\mathbf{X}}_n$. To deduce the linearity of its energy profile, the full force of asymptotic independence is not needed; Lemma 4.1 alone suffices.

5. DEDUCING RESULTS FOR $\mathbf{X}_t$ FROM THOSE FOR $\hat{\mathbf{X}}_n$

Proof of Proposition 1.2: Let $\hat{\mu}$ be the unique invariant measure for $\hat{\mathbf{X}}_n$ given by Lemma 2.1, and assume, for the moment, that $Z := \int \frac{1}{\sqrt{\eta}}\, d\hat{\mu} < \infty$. Since for $\mathbf{X}_t$, the expected holding time at $(\alpha, \eta, \xi_1, \ldots, \xi_N)$ is $\frac{1}{\sqrt{\eta}}$, we conclude from standard arguments that $\mu := \frac{1}{Z}\frac{1}{\sqrt{\eta}}\hat{\mu}$ is the unique invariant measure for $\mathbf{X}_t$. In more detail, let G be a continuous, compactly supported function on $(\alpha, \eta, \xi_1, \ldots, \xi_N)$-space, and consider a typical sample path $\hat{X}_n = (\alpha_n, \eta_n, \xi_{1,n}, \ldots, \xi_{N,n})$ of $\hat{\mathbf{X}}_n$, with typical holding times τ_n (which are independent with exponential law, mean $\frac{1}{\sqrt{\eta_n}}$). The ergodic theorem applied to $(\hat{\mathbf{X}}_n, \hat{\mu})$ guarantees that as $k \to \infty$,

$$\frac{1}{k}\sum_{n=0}^{k-1} \tau_n \;\to\; \int \frac{1}{\sqrt{\eta}} d\hat{\mu} \quad \text{and} \quad \frac{1}{k}\sum_{n=0}^{k-1} \tau_n G(\hat{X}_n) \;\to\; \int \frac{1}{\sqrt{\eta}} G d\hat{\mu}\ .$$

Let X_t be the sample path of $\mathbf{X}_t$ corresponding to $\hat{X}_n$ and τ_n, and let $T(k) = \sum_{n=0}^{k-1} \tau_n$. Then

$$\lim_{k\to\infty} \frac{1}{T(k)} \int_0^{T(k)} G(X_t) dt = \lim_{k\to\infty} \frac{k}{T} \cdot \frac{1}{k}\sum_{n=0}^{k-1} \tau_n G(\hat{X}_n) = \frac{1}{Z} \cdot \int \frac{1}{\sqrt{\eta}} G d\hat{\mu}.$$

To show $Z < \infty$, we use again the coupling argument in the proof of Lemma 1.2, this time setting both bath temperatures equal to T_L for $\hat{\mathbf{X}}'_n$ (assuming $T_L < T_R$ as before). Then $\hat{\mu}' \le \hat{\mu}$, and since $\int \frac{1}{\sqrt{\eta'}} d\hat{\mu}' < \infty$ (an easy exercise), it follows that $Z < \infty$. □

Let $x \in (0, 1)$ and $m_{-\ell}, \ldots, m_\ell \in \mathbb{Z}^+ \cup \{0\}$ be fixed, and let $G^{(N)} = \prod_{i=-\ell}^{\ell} \xi_{[xN]+i}^{m_i}$.

Lemma 5.1.

$$\lim_{N\to\infty} \int \frac{1}{\sqrt{\eta}} G^{(N)} d\hat{\mu}^{(N)} \;=\; \lim_{N\to\infty} \left(\int \frac{1}{\sqrt{\eta}} d\hat{\mu}^{(N)} \int G^{(N)} d\hat{\mu}^{(N)} \right).$$

Proof of Theorem 1 and Corollary 1 Assuming Lemma 5.1: First, we have

$$\begin{aligned}\lim_{N\to\infty}\int G^{(N)}d\mu^{(N)} &= \lim_{N\to\infty}\frac{1}{Z^{(N)}}\int G^{(N)}\frac{1}{\sqrt{\eta}}d\hat{\mu}^{(N)}\\ &= \lim_{N\to\infty}\frac{1}{Z^{(N)}}\int\frac{1}{\sqrt{\eta}}d\hat{\mu}^{(N)}\int G^{(N)}d\hat{\mu}^{(N)}\\ &= \lim_{N\to\infty}\int G^{(N)}d\hat{\mu}^{(N)}.\end{aligned}$$

The first equality expresses how μ is related to $\hat{\mu}$ (Proposition 1.2), the second is by Lemma 5.1, and the third is the definition of Z. By Propositions 2.1 and 2.2,

$$\lim_{N\to\infty}\int G^{(N)}d\hat{\mu}^{(N)} \;=\; \left(\Pi_{i=\ell}^{\ell}m_i!\right)T^{m_{-\ell}+\cdots+m_\ell}$$

where $T=(1-x)T_L+xT_R$. Since this holds for all x and all $m_{-\ell},\ldots,m_\ell$, LTE and linearity of profile for $\mathbf{X}_t$ is proved. □

Proof of Lemma 5.1: First, in the notation of Sec. 3, for fixed N and i,

$$E_{\hat{\mu}^{(N)}}[F(\cdot,\underline{k}^*)|\alpha=i] \;=\; \int T_L^{k_L}T_R^{k_R}dv_{\underline{k}^*,i}.$$

This is because requiring $\alpha=i$ in the last step when running $\hat{\mathbf{X}}_n$ corresponds to running $\mathbf{Y}_n$ with $\gamma_0=i$.

Next we fix x, ℓ and y, and let $\hat{\mu}^{(N)}_{x,\ell|y}$ denote the projection of $\hat{\mu}^{(N)}|\{\alpha=[yN]\}$ onto $(\eta,\xi_{[xN]-\ell},\ldots,\xi_{[xN]+\ell})$-space. Our results in Secs. 3 and 4 applied to initial conditions with two groups of particles (Case (2) under "Setting" in Sec. 4.2) give, for $x\neq y$,

$$\hat{\mu}_{x,\ell|y} \;:=\; \lim_{N\to\infty}\hat{\mu}^{(N)}_{x,\ell|y} \;= d\eta\beta(y)\mathrm{e}^{-\beta(y)\eta}\times\Pi_{i=-\ell}^{\ell}\,d\xi_{[xN]+i}\;\beta(x)\mathrm{e}^{-\beta(x)\xi_{[xN]+i}}$$

with $\beta(y)^{-1}=(1-y)T_L+yT_R$ and $\beta(x)^{-1}=(1-x)T_L+xT_R$.

Finally, let $G^{(N)}$ be as in Lemma 5.1. Our line of argument (with justification to follow) is

$$\begin{aligned}\lim_{N\to\infty}E_{\hat{\mu}^{(N)}}\left[\frac{1}{\sqrt{\eta}}G^{(N)}\right] &= \lim_{N\to\infty}\int_0^1 E_{\hat{\mu}^{(N)}_{x,\ell|y}}\left[\frac{1}{\sqrt{\eta}}G^{(N)}\right]dy\\ &= \int_0^1\left(\lim_{N\to\infty}E_{\hat{\mu}^{(N)}_{x,\ell|y}}\left[\frac{1}{\sqrt{\eta}}G^{(N)}\right]\right)dy\\ &= \int_0^1 E_{\hat{\mu}_{x,\ell|y}}\left[\frac{1}{\sqrt{\eta}}G^{(N)}\right]dy\end{aligned}$$

$$= \int_0^1 E_{\hat{\mu}_{x,\ell|y}} \left[\frac{1}{\sqrt{\eta}} \right] \cdot E_{\hat{\mu}_{x,\ell|y}} \left[G^{(N)} \right] \, dy$$

$$= \lim_{N \to \infty} E_{\hat{\mu}^{(N)}} \left[\frac{1}{\sqrt{\eta}} \right] \cdot E_{\hat{\mu}^{(N)}} \left[G^{(N)} \right].$$

Exchanging the order of the limit and integral in the second equality is justified because we have dominated convergence: By Hölder's inequality,

$$\int \frac{1}{\sqrt{\eta}} G^{(N)} d\mu_{x,\ell|y}^{(N)} \leq \left\{ \int \frac{1}{\eta^{\frac{3}{4}}} d\mu_{x,\ell|y}^{(N)} \right\}^{\frac{2}{3}} \cdot \left\{ \int \left(G^{(N)} \right)^3 d\mu_{x,\ell|y}^{(N)} \right\}^{\frac{1}{3}}.$$

The first factor on the right is bounded above by comparing it to the corresponding integral for a process with both bath temperatures equal to T_L (assuming $T_L < T_R$). The second factor is estimated similarly by comparing to a process with bath temperatures equal to T_R. The next two equalities are based on the existence of the limit $\hat{\mu}_{x,\ell|y}$ and its product structure from Sec. 4, and the last equality is obtained by reversing the argument for the two expectations separately. □

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